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Quantum Mechanics in Dirac's Front Form

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In 1949 Dirac published a paper in which he proposed various ways to combine special relativity with the Hamiltonian formulation of dynamics; these were referred to as *forms* and three different *forms*, the *instant*, the *point*, and the *front forms*, were discussed. Dirac considered the *front form* to be "mathematically the most interesting." Despite this, the *front form* appears to have been the least explored. This paper presents the results of a study of quantum mechanics in the *front form*.

1. INTRODUCTION

In 1949 Dirac published a paper entitled Forms of Relativistic Dynamics in which he proposed various ways to combine special relativity with the Hamiltonian formulation of dynamics; these were referred to as *forms* and three different *forms*, the *instant*, the *point*, and the *front forms*, were discussed (Dirac, 1949). That generally adopted is the *instant form*, which is based on physical data at some instant of time relative to a particular inertial observer, i.e., on a constant-time spacelike hypersurface. The *point form* is based on an observer's past lightcone. A detailed formulation of classical mechanics on an observer's past lightcone as well as an attempt at quantization have been given by Derrick (1987*a,b*). These ideas have been pursued more recently by Mosley and Farina (1992*a,b*).

The last of these *forms* was based on the 3-dimensional hypersurface, hereafter called the light front, formed by a plane wave propagating with the velocity of light; Dirac called this the *front form*. The idea was to set up a theory in which the dynamical variables referred to physical conditions on a light front. This *form*, which was considered to be "mathematically the most interesting" by Dirac (1949), has been less explored than the

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others, although light-front field theories have been discussed (Chang *et al.*, 1973; Pauli and Brodsky, 1985; Harindranath and Vary, 1987). Our objective is to establish quantum mechanics in the *front form.*

2. LIGHT-FRONT COORDINATES

For clarity we shall start by considering a $(1+1)$-dimensional space-time denoted by $\mathscr{M}^2$, i.e., we consider a classical particle moving in one dimension. Let $x^0 = ct$ and $x^1 = x$ be coordinates associated with an inertial observer, i.e., with respect to these coordinates the metric has components given by the diagonal matrix $\eta_{\mu\nu} = (1, -1)$; we shall call these Cartesian coordinates. In the *instant form* physical data at time t are specified on the hypersurface $\mathscr{S}_t$ of constant time t. $\mathscr{S}_t$ is the usual spatial space of the observer at time t; the symbol $\mathscr{S}_t$ is introduced here in place of the usual notation $\mathbb{R}$ for the spatial space. The symbol $\mathbb{R}$ will be used to denote the real line without attachment to space-time. Following Dirac, surfaces of constant values of $ct + x$ are called light fronts. We can introduce light-front coordinates y^μ related to x^ν by

$$y^0 = ct + x, \qquad y^1 = x$$

It is convenient to introduce the variable $\tau = y^0/c$, which has the dimension of time, and write y^1 simply as y as we did for x^1. We then have a family of light fronts $\mathscr{F}_\tau$ (parametrized by τ) covering the entire space-time $\mathscr{M}^2$. Points on $\mathscr{F}_\tau$ are specifiable by coordinates $(c\tau, y)$ which are not Cartesian since the components of the metric are now given by the matrix $g_{\mu\nu}$, where

$$g_{\mu\nu} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

3. CLASSICAL MOTION IN LIGHT-FRONT COORDINATES IN $\mathscr{M}^2$

Traditionally an inertial observer describes the motion of a classical particle of rest mass m_0 in his or her Cartesian coordinates (ct, x); the observer is supposed to know the actual position x of the particle at each time t. A free particle moving with speed $v = dx/dt$ will have the usual expressions for the canonical momentum and Hamiltonian

$$p = mv; \quad H = (p^2c^2 + m_0^2c^4)^{1/2}; \quad \text{where } m = m_0/(1 - v_c^2)^{1/2} \text{ and } v_c = v/c$$

derived from the Lorentz-invariant variational principle

$$\delta\left\{-m_0 c \int [\eta_{\mu\nu}\, dx^\mu\, dx^\nu]^{1/2}\right\} = 0$$

with Lagrangian

$$-m_0c^2[\eta_{\mu\nu}(dx^\mu/dx^0)(dx^\nu/dx^0)]^{1/2}$$

To describe the motion of the particle on $\mathscr{F}_\tau$ with coordinates y^μ we can employ a Hamiltonian formulation derived from the variational principle

$$\delta\left\{-m_0c\int[g_{\mu\nu}(dy^\mu)(dy^\nu)]^{1/2}\right\}=0$$

with Lagrangian

$$L=-m_0c^2[g_{\mu\nu}(dy^\mu/dy^0)(dy^\nu/dy^0)]^{1/2}=-m_0c^2[1-2c^{-1}(dy/d\tau)]^{1/2}$$

The canonical momentum π is defined by $\pi=\partial L/\partial w$, where $w=dy/d\tau$, and the Hamiltonian by $K=w\pi-L$. We have

$$\pi=\frac{-m_0^2c^3}{L};\qquad K=\frac{1}{2}c\left(\pi+\frac{m_0^2c^2}{\pi}\right)$$

The following properties are obvious:

1. $\pi>0$, irrespective of the direction of motion (sign of w). As will be seen later, this result imposes a severe constraint on the way this theory is quantized.
2. A stationary particle, i.e., with $w=0$, has a nonzero momentum, i.e., $\pi_0=m_0c$.
3. The Hamiltonian K is positive definite with the value m_0c^2 for a stationary particle as expected.

For the motion of a free particle in $\mathscr{M}^2$ we have

$$w_c=\frac{1}{2}\left(1-\frac{(m_0c)^2}{\pi^2}\right),\qquad \pi=\frac{m_0c}{(1-2w_c)^{1/2}},\qquad y=w(\tau-\tau_0)+y_0$$

Here w_c denotes the ratio w/c and takes values in $(-\infty,\frac{1}{2})$; π takes the corresponding values in $(0,\infty)$ with $\pi=\pi_0=m_0c$ when $w_c=0$. Note that the velocity w_c is related to v_c by $w_c=v_c/(1+v_c)$; $\pi<\pi_0$ corresponds to negative w_c, which can assume arbitrarily large negative values as v tends to $-c$.

4. CANONICAL TRANSFORMATIONS RELATING QUANTITIES IN CARTESIAN AND IN LIGHT-FRONT COORDINATES

Canonical variables (y,π) and (q,p) in light-front and Cartesian coordinates (Appendix A) can be related by a canonical transformation obtained from an appropriate generating function $f(p,y)$ through the relations

$$\pi=-\partial f/\partial y,\qquad q=-\partial f/\partial p$$

On $\mathcal{M}^2$ the required generating function is

$$f(p, y) = -(p_0 + p)y, \qquad \text{where} \quad p_0 = (p^2 + m_0^2 c^2)^{1/2}$$

This leads to the transformation

$$y = \frac{p_0}{p_0 + p} q, \qquad \pi = p_0 + p, \qquad K = H = \frac{c(\pi^2 + m_0^2 c^2)}{2\pi}$$

Equivalently with the generating function

$$f'(\pi, q) = -\frac{\pi^2 - m_0^2 c^2}{2\pi} q \qquad \text{with} \quad p = -\frac{\partial f'}{\partial q}, \qquad y = -\frac{\partial f'}{\partial \pi}$$

we obtain the inverse transformation

$$q = \frac{2\pi^2}{\pi^2 + m_0^2 c^2} y, \qquad p = \frac{\pi^2 - m_0^2 c^2}{2\pi}, \qquad H = K = (p^2 c^2 + m_0^2 c^4)^{1/2}$$

Finally we should point out that the choice of light fronts is not unique. We could have chosen the light fronts given by constant values of $ct - x$, i.e., we can introduce light-front coordinates y'^{μ} related to x^{μ} by

$$y'^0 = ct - x, \qquad y'^1 = x$$

We then have a family of light fronts $\mathcal{F}_\tau^-$ which also covers the entire space-time. The components of the metric are now given by the matrix $g'_{\mu\nu}$ where

$$g'_{\mu\nu} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

5. QUANTUM MECHANICS IN $\mathcal{M}^2$ IN THE FRONT FORM

The object here is to obtain a quantum theory in terms of variables on $\mathcal{F}_\tau$. We shall first establish a quantum Hilbert space on $\mathcal{F}_\tau$.

5.1. Hilbert Spaces on Light Front

Let $\xi(x^0, x)$ be a complex scalar function on $\mathcal{M}^2$; then the integral on $\mathcal{M}^2$ given by

$$\iint dx^0\, dx |\xi(x^0, x)|^2 \delta((c\tau - x^0)^2 - x^2)$$
$$= \int \frac{dx}{2|x|} (|\xi(c\tau - x, x)|^2 + |\xi(c\tau + x, x)|^2)$$

is Lorentz invariant, assuming the integral exists. The δ-function reduces the two-dimensional integral to two invariant one-dimensional integrals the first of which is over the light front $\mathscr{F}_\tau$ and the other over $\mathscr{F}_\tau^-$. Introduce functions $\Phi_\tau(y)$ on $\mathscr{F}_\tau$ by $\Phi_\tau(y) = \xi(c\tau - y, y)$; then Φ as a function of the light-front coordinates $(c\tau, y)$ is the restriction of ξ on $\mathscr{M}^2$ to the light front $\mathscr{F}_\tau$. We shall use these functions on $\mathscr{F}_\tau$ to construct a Hilbert space $\mathscr{H}(\mathscr{F}_\tau)$ for the purpose of establishing a quantum mechanics on $\mathscr{F}_\tau$ (Schweber, 1964). More precisely we take

$$\mathscr{H}(\mathscr{F}_\tau) = L^2(\mathscr{F}_\tau, d\mu)$$

where square-integrability is with respect to the measure $d\mu = dy/|y|$. Note that $L^2(\mathscr{F}_\tau, d\mu)$ is unitarily equivalent to $L^2(\mathscr{F}_\tau, dy)$ with the unitary map V which maps $\phi_\tau \in L^2(\mathscr{F}_\tau, dy)$ to $\Phi_\tau \in L^2(\mathscr{F}_\tau, d\mu)$ given by

$$\Phi_\tau(y) = (V\phi_\tau)(y) = |y|^{1/2}\phi_\tau(y), \qquad \phi_\tau(y) = (V^{-1}\Phi_\tau)(y) = \Phi_\tau(y)/|y|^{1/2}$$

So, we can identify $\mathscr{H}(\mathscr{F}_\tau)$ with the more familiar $L^2(\mathscr{F}_\tau, dy)$ from now on. For brevity we shall write $\phi_\tau(y), \Phi_\tau(y)$ as $\phi(y), \Phi(y)$, suppressing the subscript τ.

5.2. Quantization on the Light Front

Traditionally, as with classical mechanics, quantum theory is in the *instant form* since it is formulated in terms of vectors and operators in the Hilbert space $\mathscr{H}(\mathscr{S}_t) = L^2(\mathscr{S}_t, dx)$ (functions on $\mathscr{S}_t$ square-integrable with respect to the measure dx). To establish a quantum theory in the *front form* we shall adopt, initially, the Hilbert space $\mathscr{H}(\mathscr{F}_\tau)$ and quantize the basic classical observables π, H, and y. A more detailed analysis will subsequently reveal that $\mathscr{H}(\mathscr{F}_\tau)$ is not the appropriate Hilbert space because we are dealing with quantization under a constraint which usually leads to many complications (Dirac, 1966).

Let us consider operators in the Hilbert space $L^2(\mathscr{F}_\tau, dy)$ first; the corresponding operators in $\mathscr{H}(\mathscr{F}_\tau)$ can be obtained by a unitary transformation.

A first attempt might be to associate π with the self-adjoint operator $\hat{k} = -i\hbar\, d/dy$ in $L^2(\mathscr{F}_\tau, dy)$. A difficulty arises from the constraint that the classical observable π is positive definite. We can accommodate this constraint in the usual manner, i.e., by assuming that physical states are described only by members of the subspace of $L^2(\mathscr{F}_\tau, dy)$ which corresponds to the positive part of the operator $\hat{k}$. To formalize these ideas we shall introduce some related quantities.

Elements ϕ of $L^2(\mathscr{F}_\tau, dy)$ are functions of y, i.e., $\phi = \phi(y)$. The Fourier transform of ϕ is

$$\tilde{\phi}(k) = \tilde{F}\phi = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dy\, \phi(y) \exp(-\not{i}ky), \qquad \not{i} = i/\hbar$$

which can be regarded as a function of the variable $k \in \mathbb{R}$, i.e., a function of "k-space" $\tilde{\mathscr{F}}_\tau = \{k : k \in \mathbb{R}\}$. Here $\tilde{F}$ denotes the Fourier transform operator on $L^2(\mathscr{F}_\tau, dy)$.

Being square-integrable with respect to the measure dk, these transformed functions $\tilde{\phi}(k)$ form a Hilbert space to be denoted by $\tilde{L}^2(\tilde{\mathscr{F}}_\tau, dk)$, i.e., $\tilde{L}^2(\tilde{\mathscr{F}}_\tau, dk) = \tilde{F}L^2(\mathscr{F}_\tau, dy)$. Let $L^2_+(\mathscr{F}_\tau, dy)$ be the subspace of $L^2(\mathscr{F}_\tau, dy)$ which corresponds to the positive part of the operator $\hat{k}$. Since the Fourier transform $\tilde{k} = \tilde{F}\hat{k}\tilde{F}^{-1}$ of the operator $\hat{k}$ is the multiplication operator k in $\tilde{L}^2(\tilde{\mathscr{F}}_\tau, dk)$ it follows that $L^2_+(\mathscr{F}_\tau, dy)$ consists of elements ϕ whose Fourier transforms $\tilde{\phi}$ are functions of the variable k of the form $\tilde{\phi}(k) = 0$ for almost every $k < 0$; functions in $L^2_+(\mathscr{F}_\tau, dy)$ which correspond to such truncated Fourier transforms are known as Hardy class functions (Bohm and Gadella, 1989). These functions form a subspace since $\tilde{L}^2_+(\tilde{\mathscr{F}}_\tau, dk) = \tilde{F}L^2_+(\mathscr{F}_\tau, dy)$ is obviously a subspace of $\tilde{L}^2(\tilde{\mathscr{F}}_\tau, dk)$. Note that a subspace always means a closed subspace. Furthermore $L^2_+(\mathscr{F}_\tau, dy)$ clearly reduces the operator $\hat{k}$. We shall denote the reduction of $\hat{k}$ in $L^2_+(\mathscr{F}_\tau, dy)$ by $\hat{k}_+$ and denote the domain of $\hat{k}_+$ by $\mathscr{D}_{\hat{k}_+}$. In the space $\mathscr{H}(\mathscr{F}_\tau)$ we have the corresponding subspace $\mathscr{H}(\mathscr{F}_\tau)^+ = VL^2_+(\mathscr{F}_\tau, dy)$. Let $\hat{R}^+$ denote the projector from $\mathscr{H}(\mathscr{F}_\tau)$ onto $\mathscr{H}(\mathscr{F}_\tau)^+$, i.e., $\hat{R}^+\mathscr{H}(\mathscr{F}_\tau) = \mathscr{H}(\mathscr{F}_\tau)^+$, and let $\hat{\pi}$ denote the operator $V\hat{k}_+V^{-1}$, which is clearly self-adjoint in $\mathscr{H}(\mathscr{F}_\tau)^+$.

We are now in a position to present a quantization scheme based on geometric quantization (Woodhouse, 1980; Wan and Viasminsky, 1977; Wan *et al.*, 1984*b*):

1. Every quantum system is associated with a Hilbert space, called the quantum Hilbert space of the system. Pure states correspond to unit vectors in the Hilbert space and observables correspond to positive operator-valued measures.
2. In the case of a free particle in the *front form* the quantum Hilbert space is $\mathscr{H}(\mathscr{F}_\tau)^+$.
3. The momentum π is quantized as the self-adjoint operator $\hat{\pi}$ in $\mathscr{H}(\mathscr{F}_\tau)^+$ where
$$\hat{\pi} = V\hat{k}_+V^{-1} = -i\hbar|y|^{1/2}\frac{d}{dy}\frac{1}{|y|^{1/2}}$$
acting on the domain $\mathscr{D}_{\hat{\pi}} = V\mathscr{D}_{\hat{k}_+} \in \mathscr{H}(\mathscr{F}_\tau)^+$.
4. The Hamiltonian K is quantized as the self-adjoint operator $\hat{K} = \frac{1}{2}c(\hat{\pi} + m_0^2c^2/\hat{\pi})$ in $\mathscr{H}(\mathscr{F}_\tau)^+$.
5. The position y is quantized as the multiplication operator $\hat{y} = y$ in $\mathscr{H}(\mathscr{F}_\tau)^+$.

This quantized theory possesses a number of features that distinguish it from conventional quantum mechanics:

1. We have adopted a generalized quantum mechanics (GQM) which admits positive operator-valued (POV) measures as observables. Various studies on the foundations of quantum mechanics have strongly suggested that POV measures should be admitted as observables (Kraus, 1983; Busch *et al.*, 1991). Mathematically, projector-valued (PV) measures correspond one-to-one to self-adjoint operators, while POV measures correspond to symmetric operators. It follows that the generalized quantum theory admits symmetric operators as observables. This generalization of quantum theory opens up a whole new avenue for dealing with quantization problems. When quantizing a classical observable within orthodox quantum theory one would insist on obtaining a self-adjoint operator as the quantized observable; failing this, one would say that the classical observable cannot be quantized (Wan *et al.*, 1984*a*). A quantization scheme adapted to GQM need only to produce symmetric operators. Accepting maximal symmetric operators as observables would not lead to a radical departure from traditional quantum theory, since maximal symmetric operators correspond one-to-one to POV measures in the same way that self-adjoint operators correspond to PV measures (Akhiezer and Glazman, 1963, p. 135). The relevance of this discussion of GQM will become obvious when we come to look at the light-front position observable.

2. The quantum Hilbert space is taken to be the subspace $\mathscr{H}(\mathscr{F}_\tau)^+$ of $\mathscr{H}(\mathscr{F}_\tau)$. Elements $\Phi(y)$ of $\mathscr{H}(\mathscr{F}_\tau)^+$ are related to Hardy class functions. It follows that states are described by wave functions $\Phi(y)$ which can vanish only over a set of measure zero, e.g., $\Phi(y)$ cannot be equal to zero over any nonvanishing interval. In other words, there are no "localized" wave functions.

3. The momentum operator $\hat{\pi}$ possesses an inverse, since it does not admit zero as an eigenvalue. The Hamiltonian operator $\hat{K}$ is therefore well defined, self-adjoint, and positive; $\hat{K}$ has a simple spectrum (Jauch, 1968, p. 47) lower bounded by m_0c^2. This follows from the positive nature of the momentum $\hat{\pi}$, which is consistent with the classical constraint $\pi > 0$. If one had not taken this classical constraint into account, one would have obtained a Hamiltonian operator which is not positive. We should mention that $\hat{\pi}$ is identical to the restriction to $\mathscr{H}(\mathscr{F}_\tau)^+$ of the operator in $\mathscr{H}(\mathscr{F}_\tau)$ obtained by quantizing π geometrically. Note that in applying the geometric quantization formula (Wan and Viasminsky, 1977) the square root of the determinant of the metric there should be taken to be $|y|^{-1}$.

4. The existence or otherwise of a position observable in relativistic quantum mechanics has long been a source of controversy; it is certainly not a routine matter to find a position observable (Kalnay, 1971). Our present theory is no exception. The position operator $\hat{y}$ is not self-adjoint, but only maximal symmetric in $\mathscr{H}(\mathscr{F}_\tau)^+$ (see Appendix B). This is closely

related to the nonlocalizable nature of the wave function $\Phi(y)$ discussed above. Orthodox quantum theory with its insistence on self-adjointness of all its observables would rule $\hat{y}$ out as an observable. This is why we adopt the generalized quantum theory according to (1) above. The POV measure of $\hat{y}$, denoted by $\hat{P}(\hat{y}; b)$, is given by $\hat{R}^+\chi_b(y)\hat{R}^+$ (see Appendix C). Physically, having $\hat{y}$ as the position observable and $\mathscr{H}(\mathscr{F}_\tau)^+$ as the quantum Hilbert space means that the particle cannot be sharply localized, i.e., the probability of the particle being in any finite interval b given by $\|\hat{P}(\hat{y}; b)\Phi\|^2$ is always less than 1. We should point out that this is not the first time that a POV measure has been used to represent a position observable. Kraus (1983) argued strongly that POV measures should be used to represent position observables of photons.

Relativistic quantum mechanics in the *instant form* encounters a difficulty in the form of Hegerfeldt's theorem (Hegerfeldt, 1974), which states that "In a relativistic quantum theory of point particles there are no one-particle states that are ever localized in a finite spatial region and yet obey at later times the principle of relativistic causality" (Prugovecki, 1986, p. 82). This theorem is based on the mathematical result that unitary time evolution generated by a positive Hamiltonian operator leads to an instant spreading of the wave packet. Clearly the present theory suffers no such difficulty, because there are no states in the quantum Hilbert space $\mathscr{H}(\mathscr{F}_\tau)^+$ which are localized in a finite spatial region in y at any instant.

In adopting $\mathscr{H}(\mathscr{F}_\tau)^+$ as the quantum Hilbert space our quantization scheme is now seen to give a quantum momentum observable $\hat{\pi}$ consistent with the classical constraint, and a position observable $\hat{y}$ which makes the above Hegerfeldt theorem irrelevant.

Finally, we should mention that there exists a continuity equation relating the probability density ρ_0 and the probability current density ρ_1 (Appendix D).

5. Geometric quantization provides a unitary mapping between this light-front quantum mechanics and the orthodox *instant form*. This will be demonstrated in the next subsection.

5.3. Classical Canonical Transformations and Quantum Unitary Transformations

Classically the *instant form* and the *front form* are related by a canonical transformation. A similar situation also exists quantum mechanically, i.e., there is a unitary map of the Hilbert space $\mathscr{H}(\mathscr{F}_\tau)^+$ onto $\mathscr{H}(\mathscr{S}_t) = L^2(\mathscr{S}_t, dq)$ with $\tau = t$ as in the classical canonical transformation. For technical reasons it turns out to be more convenient, in the first instance, to set up a mapping between $\mathscr{H}(\mathscr{F}_\tau)^+$ and the Cartesian momentum-space wave functions. Let $\tilde{\mathscr{S}}_t$ be the momentum space conjugate to $\mathscr{S}_t$, and $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t) = \tilde{L}^2(\tilde{\mathscr{S}}_t, dp)$ be the Hilbert space of functions $\tilde{\psi}_t(p)$ on $\tilde{\mathscr{S}}_t$ square-integrable with respect to the measure dp. As before we shall drop the subscript t from the wave function from now on. $\tilde{L}^2(\tilde{\mathscr{S}}_t, dp)$ is of course the Fourier transform space of $L^2(\mathscr{S}_t, dq)$. We shall demonstrate explicitly in what follows that a unitary mapping exists between $\mathscr{H}(\mathscr{F}_\tau)^+$ and $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t)$. For later convenience we shall call $\mathscr{H}(\mathscr{F}_\tau)^+$ and $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t)$ the light-front coordinate representation and Cartesian momentum representation, respectively.

5.3.1. *Unitary Transformations of Hilbert Spaces*

In geometric quantization a prescription known as pairing is given to relate observables quantized in different coordinate systems whose associated classical canonical variables (q', p') and (q'', p'') are related by a canonical transformation defined by a generating function $f_0(p', q'')$ (Woodhouse, 1980). The pairing defines a map T which relates the wave functions $\varphi''(q'')$ to wave functions $\tilde{\varphi}'(p')$. T is explicitly given by

$$\tilde{\varphi}'(p') = (T\varphi'')(p') = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \varphi''(q'') \exp(\not{i} f_0) \left|\frac{\partial^2 f_0}{\partial p' \partial q''}\right|^{1/2} g''(q'')^{1/4}\, dq''$$

with the inverse given by

$$\varphi''(q'') = (T^{-1}\varphi')(q'')$$
$$= \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \tilde{\varphi}'(p') \exp(-\not{i} f_0) \left|\frac{\partial^2 f_0}{\partial p' \partial q''}\right|^{1/2} g''(q'')^{-1/4}\, dp'$$

Here $g''(q'')$ is the determinant of the metric in coordinates q''.

The pairing construction does not always lead to unitary transformations; we have to check each case carefully. Applying this pairing construction to our present case, we have the following map between $\mathscr{H}(\mathscr{F}_\tau)^+$ and $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t)$ with $\tau = t$:

$$\tilde{\psi}(p) = (T\Phi)(p) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \Phi(y) \exp[-\not{i}(p + p_0)y] \left(\frac{p + p_0}{p_0}\right)^{1/2} \frac{dy}{|y|^{1/2}}$$

First we observe that the integral exists, since it is the Fourier transform of $\Phi(y)/|y|^{1/2}$, which is square-integrable with respect to the measure dy. The map preserves the scalar product and is bijective with the inverse map given by (Appendix E)

$$\Phi(y) = (T^{-1}\tilde{\psi})(y) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \tilde{\psi}(p) \exp[\not{i}(p + p_0)y] \left(\frac{p + p_0}{p_0}\right)^{1/2} |y|^{1/2}\, dp$$

We have therefore established the existence of a unitary map between the

Hilbert spaces $\mathscr{H}(\mathscr{F}_\tau)^+$ and $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t)$, and hence between $\mathscr{H}(\mathscr{F}_\tau)^+$ and $\mathscr{H}(\mathscr{S}_t)$.

The unitary map T enables us to transform operators between the light-front and the Cartesian pictures as described in the next subsection.

5.3.2. Unitary Transformations of Operators

First we can express operators in light-front coordinate representation in terms of Cartesian momentum representation:

- The momentum observable $\hat{\pi}$: $T\hat{\pi}T^{-1} = p + p_0$.
- The Hamiltonian $\hat{K}$: $T\hat{K}T^{-1} = (p^2c^2 + m_0^2c^4)^{1/2}$.
- The position observable $\hat{y}$:

$$T\hat{y}T^{-1} = i\hbar\left\{\left(\frac{p_0}{p_0 + p}\right)\frac{d}{dp} + \frac{p - p_0}{2(p + p_0)p_0}\right\}$$

Let $\tilde{H} = (p^2c^2 + m_0^2c^4)^{1/2}$; then $\tilde{H}$ is clearly the Fourier transform of the quantized operator $\hat{H}$ in the Cartesian coordinate representation space $\mathscr{H}(\mathscr{S}_t)$, i.e., $\hat{H} = (\hat{p}^2c^2 + m_0^2c^4)^{1/2}$, where $\hat{p} = -i\hbar\, d/dq$.

It is interesting to note that for the position observable the same result is obtained by quantizing the classical expression $y = [p_0/(p_0 + p)]q$ using the symmetrization rule, i.e.,

$$T\hat{y}T^{-1} = \frac{1}{2}\left\{\left(\frac{p_0}{p_0 + p}\right)\tilde{q} + \tilde{q}\left(\frac{p_0}{p_0 + p}\right)\right\}$$

where $\tilde{q} = i\hbar\, \partial/\partial p$ is the position operator in the momentum representation space $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t)$. The operator $T\hat{y}T^{-1}$ is of course not self-adjoint. This is consistent with geometric quantization since the vector field generated by the function $y = \{p_0/(p_0 + p)\}q$ is incomplete, so it cannot be quantized to give a self-adjoint operator. As pointed out earlier, such an operator can still represent an observable in a generalized quantum theory. The expressions for these operators in the coordinate representation, i.e., in $\mathscr{H}(\mathscr{S}_t)$, can be obtained by a Fourier transformation and for brevity will not be given here.

Conversely we can also express operators in Cartesian momentum representation in terms of light-front coordinate representation:

- The momentum $\hat{p}$: $T^{-1}\hat{p}T = (\hat{\pi}^2 - m_0^2)/2\hat{\pi}$.
- The Hamiltonian $\hat{H}$: $T^{-1}\hat{H}T = c(\hat{\pi}^2 + m_0^2c^2)/2\hat{\pi} = \hat{K}$.
- The position $\hat{q}$: $T^{-1}\hat{q}T = \frac{1}{2}\{[\hat{\pi}^2/(\hat{\pi}^2 + m_0^2c^2)]y + y[\hat{\pi}^2/(\hat{\pi}^2 + m_0^2c^2)]\}$.

Finally we should mention that the domains of various operators are related by the unitary operator T in the standard fashion.

6. QUANTUM MECHANICS IN $\mathscr{M}^4$ IN THE FRONT FORM

We can extend our analysis to four-dimensional spacetime $\mathscr{M}^4$. The results are summarized in what follows. The light-front coordinates y^μ are related to Cartesian coordinates x^μ by

$$y^0 = x^0 + x^1, \qquad y^j = x^j \qquad \text{for} \quad j = 1, 2, 3$$

A three-dimensional hypersurface of constant y^0 is a light front in $\mathscr{M}^4$. This light-front will be denoted by $\mathscr{F}_\tau$ with $\tau = y^0/c$, while a constant-time $t = \tau/c$ hypersurface will be denoted by $\mathscr{S}_t$. The metric in light-front coordinates has components

$$g_{\mu\nu} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Classical motion for a free particle is described by the Lagrangian

$$L = -m_0 c^2 [1 - 2w^1/c - (w^2)^2/c^2 - (w^3)^2/c^2]^{1/2}$$

where $w^j = dy^j/d\tau$. The conjugate momentum $\pi_j = \partial L/\partial w^j$ and the Hamiltonian are given by

$$\pi_1 = \frac{-m_0^2 c^3}{L} > 0, \qquad \pi_s = \frac{-m_0^2 c^2 w^s}{L}$$

$$K = \frac{c}{2\pi_1}(|\boldsymbol{\pi}|^2 + (m_0 c)^2) = K_1 + \frac{c}{2\pi_1}(\pi_2^2 + \pi_3^2)$$

where $K_1 = \frac{1}{2} c\{\pi_1 + (m_0 c)^2/\pi_1\}$ is recognized to be the contribution along the y^1 direction. A boldface symbol denotes a three-component object, e.g., $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ with $|\boldsymbol{\pi}|^2 = \pi_1^2 + \pi_2^2 + \pi_3^2$. The index s takes the values 2, 3.

Canonical variables (y^j, π_j) on the light front $\mathscr{F}_\tau$ in light-front coordinates and (q^j, p_j) on the constant-time hypersurface $\mathscr{S}_t$ with $\tau = t$ in Cartesian coordinates (Appendix A) are related by a canonical transformation generated by the generating function

$$f(\mathbf{p}, \mathbf{y}) = -\{(p_0 + p_1)y^1 + p_2 y^2 + p_3 y^3\}, \qquad \text{where} \quad p_0 = (|\mathbf{p}|^2 + m_0^2 c^2)^{1/2}$$

which leads to the transformation

$$y^1 = \frac{p_0}{p_0 + p_1} q^1, \quad y^s = q^s - \left(\frac{p_s}{p_0 + p_1}\right) q^1, \quad \pi_1 = p_0 + p_1, \quad \pi_s = p_s$$

with the inverse transformation given by the generating function

$$f'(\boldsymbol{\pi}, \mathbf{q}) = -\left\{\left(\frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1}\right) q^1 + \pi_2 q^2 + \pi_3 q^3\right\}$$

which leads to

$$q^1 = \left(\frac{2\pi_1^2}{|\boldsymbol{\pi}|^2 + (m_0 c)^2}\right) y^1, \qquad q^s = y^s + \left(\frac{2\pi_1 \pi_s}{|\boldsymbol{\pi}|^2 + (m_0 c)^2}\right) y^1$$

$$p_1 = \frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1}, \qquad p_s = \pi_s, \qquad H = K = (|\mathbf{p}|^2 c^2 + m_0^2 c^4)^{1/2}$$

As before, we define a Hilbert space on $\mathscr{F}_\tau$ as

$$\mathscr{H}(\mathscr{F}_\tau) = L^2(\mathscr{F}_\tau, d\mu)$$

where $d\mu = d^3y/|y^1|$, introduce the corresponding map V which maps $L^2(\mathscr{F}_\tau, d\mu)$ to $L^2(\mathscr{F}_\tau, d^3y)$, and denote the Fourier transform operator on $L^2(\mathscr{F}_\tau, d^3y)$ again by $\tilde{F}$. Finally, define the subspace $\mathscr{H}(\mathscr{F}_\tau)^+$ of $\mathscr{H}(\mathscr{F}_\tau)$ consisting of those functions $\Phi \in \mathscr{H}(\mathscr{F}_\tau)$ such that $(\tilde{F}V\Phi)(\mathbf{k}) = 0$ for $k_1 < 0$.

Following the previous quantization scheme, we take $\mathscr{H}(\mathscr{F}_\tau)^+$ as the quantum Hilbert space and quantize π_j, K, and y^j as operators on $\mathscr{H}(\mathscr{F}_\tau)^+$ given by

$$\hat{y}^j = y^j, \qquad \hat{\pi}_1 = -i\hbar |y^1|^{1/2} \frac{d}{dy^1} \frac{1}{|y^1|^{1/2}}, \qquad \hat{\pi}_s = -i\hbar \frac{\partial}{\partial y^s}$$

$$\hat{K} = \frac{c}{2\hat{\pi}_1} (|\hat{\boldsymbol{\pi}}|^2 + (m_0 c)^2)$$

We can also demonstrate the existence of a unitary map of $\mathscr{H}(\mathscr{F}_\tau)^+$ onto the space $\tilde{\mathscr{H}}(\mathscr{S}_t) = \tilde{L}^2(\tilde{\mathscr{S}}_t, d^3p)$ from the pairing construction which leads to a map T between $\mathscr{H}(\mathscr{F}_\tau)^+$ and $\tilde{\mathscr{H}}(\tilde{\mathscr{S}}_t)$ given by

$$\tilde{\psi}(\mathbf{p}) = (T\Phi)(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} \exp[-\not{i}(p_0 + p_1)y^1 - \not{i}p_2 y^2 - \not{i}p_3 y^3] \times \left(\frac{p_0 + p_1}{p_0}\right)^{1/2} \frac{\Phi(\mathbf{y})}{|y^1|^{1/2}} \, d^3y$$

with the inverse map

$$\Phi(\mathbf{y}) = (T^{-1}\tilde{\psi})(\mathbf{y}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} \exp[\not{i}(p_0 + p_1)y^1 + \not{i}p_2 y^2 + \not{i}p_3 y^3] \times \left(\frac{p_0 + p_1}{p_0}\right)^{1/2} |y^1|^{1/2} \tilde{\psi}(\mathbf{p}) \, d^3p$$

Operator transformations are as follows:

$$T\hat{\pi}_1 T^{-1} = p_1 + p_0, \qquad T\hat{\pi}_s T^{-1} = p_s, \qquad T\hat{K}T^{-1} = \hat{H}$$

$$T\hat{y}^1 T^{-1} = \frac{1}{2}\left(\frac{p_0}{p_1+p_0}\tilde{q}^1 + \tilde{q}^1 \frac{p_0}{p_1+p_0}\right)$$

$$T\hat{y}^s T^{-1} = i\hbar\left\{\frac{\partial}{\partial p_s} - \left(\frac{p_s}{p_1+p_0}\right)\frac{\partial}{\partial p_1} + \frac{p_s}{2p_0(p_0+p_1)}\right\}$$

where $\tilde{q}^j = i\hbar\, \partial/\partial p_j$.

All the operators are self-adjoint except for $\hat{y}^1$, which is maximal symmetric and so is its unitary transformation (Appendix F).

We should also point out that there is a conserved current in the $3+1$ theory (Appendix D).

7. GENERATORS OF THE POINCARÉ GROUP

We shall now show how we may obtain a representation of the Poincaré algebra in terms of operators on the light front. Classically we know that the ten infinitesimal generators of the Poincaré group in Cartesian coordinates are given by

$$P_\mu = (p_1, p_2, p_3, H)$$

$$J_{ij} = q_i p_j - q_j p_i = -(q^i p_j - q^j p_i) \qquad (i, j = 1, 2, 3)$$

$$J_{0j} = -q_j H = q^j H$$

In terms of light-front coordinates we have

$$J_{23} = -\pi_3 y^2 + \pi_2 y^3$$

$$J_{31} = \pi_3 y^1 - \left(\frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1}\right) y^3$$

$$J_{12} = -\pi_2 y^1 + \left(\frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1}\right) y^2$$

$$J_{01} = c\pi_1 y^1$$

$$J_{02} = c\left(\pi_2 y^1 + \frac{\pi^2 + (m_0 c)^2}{2\pi_1} y^2\right)$$

$$J_{03} = c\left(\pi_3 y^1 + \frac{\pi^2 + (m_0 c)^2}{2\pi_1} y^3\right)$$

The expressions for P_μ have already been given in the preceding section.

We can quantize these observables geometrically in the light-front momentum space (Wan *et al.*, 1984*b*) since they all generate complete vector fields (Appendix G). Let $\tilde{\mathscr{F}}_\tau{}^+ = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ be the classical light-front momentum space, and let $\tilde{\mathscr{H}}(\tilde{\mathscr{F}}_\tau{}^+) = L^2(\tilde{\mathscr{F}}_\tau{}^+, d^3\pi)$. Then the quantized observables in $\tilde{\mathscr{H}}(\mathscr{F}_\tau{}^+)$ are all self-adjoint (Appendix G) and are given by

$$\hat{J}_{23} = i\hbar\left(\pi_2 \frac{\partial}{\partial\pi_3} - \pi_3 \frac{\partial}{\partial\pi_2}\right)$$

$$\hat{J}_{31} = i\hbar\left(\pi_3 \frac{\partial}{\partial\pi_1} - \frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1} \frac{\partial}{\partial\pi_3} + \frac{\pi_3}{2\pi_1}\right)$$

$$\hat{J}_{12} = i\hbar\left(-\pi_2 \frac{\partial}{\partial\pi_1} + \frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1} \frac{\partial}{\partial\pi_2} - \frac{\pi_2}{2\pi_1}\right)$$

$$\hat{J}_{01} = i\hbar c\left(\pi_1 \frac{\partial}{\partial\pi_1} + \frac{1}{2}\right)$$

$$\hat{J}_{02} = i\hbar c\left(\pi_2 \frac{\partial}{\partial\pi_1} + \frac{\boldsymbol{\pi}^2 + (m_0 c)^2}{2\pi_1} \frac{\partial}{\partial\pi_2} + \frac{\pi_2}{2\pi_1}\right)$$

$$\hat{J}_{03} = i\hbar c\left(\pi_3 \frac{\partial}{\partial\pi_1} + \frac{\boldsymbol{\pi}^2 + (m_0 c)^2}{2\pi_1} \frac{\partial}{\partial\pi_3} + \frac{\pi_3}{2\pi_1}\right)$$

An alternative method of obtaining these operators would be to carry out the quantization of the generators in Cartesian coordinates, and then effect a unitary transformation, using the operator T, from $\mathscr{H}(\mathscr{S}_t)$ to $\mathscr{H}(\mathscr{F}_\tau)^+$. The same results would be obtained. To verify this it is necessary to know the unitary transform relating $\tilde{\mathscr{H}}(\tilde{\mathscr{F}}_\tau{}^+)$ and $\mathscr{H}(\mathscr{F}_\tau)^+$; these are the light-front momentum and coordinate representation spaces. The pairing construction shows that the unitary transform F_+ is given by

$$(F_+ \Phi)(\boldsymbol{\pi}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} \Phi(\mathbf{y}) \exp(-i\mathbf{y} \cdot \boldsymbol{\pi}) \frac{d^3 y}{|y^1|^{1/2}}$$

with inverse

$$(F_+^{-1}\tilde{\Phi})(\mathbf{y}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{\pi_1=0}^{\infty} \int_{-\infty}^{\infty} \tilde{\Phi}(\boldsymbol{\pi}) \exp(i\mathbf{y} \cdot \boldsymbol{\pi}) \, |y^1|^{1/2} \, d^3\pi$$

The operators obtained here are different from those for the lightcone (Peres, 1967; Mosley and Farina, 1992*a*,*b*).

8. LIGHT FRONT VERSUS LIGHTCONE

Recently there has been substantial progress in developing a point-form quantum mechanics where states and other physical quantities are referred to an observer's past lightcone (Mosley and Farina, 1992*a*,*b*). It seems that most of the work on the *point form* has been of a formal nature.

We have endeavored to develop a systematic and rigorous *point-form* quantum mechanics, but have encountered severe difficulties. The root cause of all these difficulties would appear to be the singular geometry of the lightcone $\mathscr{C}$, which is not a differential manifold. To see this, consider the $(1+1)$-dimensional space-time $\mathscr{M}^2$. Introduce lightcone coordinates $\bar{y}^\mu = (c\bar{\tau}, \bar{y})$ defined in terms of the Cartesian coordinates $x^\mu = (ct, x)$ by (Derrick, 1987*a,b*)

$$\bar{y}^0 = ct + |x|, \qquad \bar{y} = x$$

In these coordinates the metric has components given by the matrix $\bar{g}_{\mu\nu}$, where

$$\bar{g}_{\mu\nu} = \begin{pmatrix} 1 & -\operatorname{sgn}(\bar{y}) \\ -\operatorname{sgn}(\bar{y}) & 0 \end{pmatrix}$$

and $\operatorname{sgn}(\bar{y})$ is the sign of $\bar{y}$. The metric is singular when $x = \bar{y} = 0$. A surface of constant $\bar{\tau}$, to be denoted by $\mathscr{C}_{\bar{\tau}}$, is a past lightcone with apex at the point $(c\tau, 0)$; we shall denote $\mathscr{C}_{\bar{\tau}}$ by $\mathscr{C}$ from now on. Even classical mechanics in the *point form* possesses some peculiar features. The Lagrangian of a free particle is given by

$$\begin{aligned} \bar{L} &= -m_0 c^2 [\bar{g}_{\mu\nu}(d\bar{y}^\mu/d\bar{y}^0)(d\bar{y}^\mu/d\bar{y}^0)]^{1/2} \\ &= -m_0 c^2 [1 - 2c^{-1} \operatorname{sgn}(\bar{y})(d\bar{y}/d\tau)]^{1/2} \end{aligned}$$

The resulting canonical momentum $\bar{\pi}$ and Hamiltonian $\bar{K}$ are given by

$$\bar{\pi} = -\frac{(m_0 c)^3 \operatorname{sgn}(\bar{y})}{L}, \qquad \bar{K} = \frac{1}{2} \operatorname{sgn}(\bar{y}) \left(\bar{\pi} + \frac{(m_0 c)^2}{\bar{\pi}} \right)$$

The nonglobal nature of the formulation is clearly seen in the fact that both $\bar{\pi}$ and $\bar{K}$ are undefined at the apex and in the discontinuous jump in $\bar{\pi}$ from a negative to a positive value as the particle passes through the origin.

Let us try to quantize the theory. Notice first that the classical system is subject to the constraint $\bar{y}\bar{\pi} > 0$, i.e., $\bar{\pi} > 0$ for $\bar{y} > 0$ and $\bar{\pi} < 0$ for $\bar{y} < 0$. As pointed out earlier, quantization under a constraint is not a straightforward matter and in fact it does not seem possible to accommodate this constraint within a *point-form* quantum mechanics. For example, suppose we begin by forming the Hilbert space $\mathscr{H}(\mathscr{C})$ of functions $\varphi(\bar{y})$ on $\mathscr{C}$ which are square-integrable with respect to the measure $d\bar{y}/|\bar{y}|$, i.e., $\mathscr{H}(\mathscr{C}) = L^2(\mathscr{C}, d\bar{y}/|\bar{y}|)$, despite the singularity of the measure at the origin. The coordinate variable $\bar{y}$ can be quantized naturally as the self-adjoint multiplication operator $\bar{y}$ in $\mathscr{H}(\mathscr{C})$, but how can we quantize $\bar{\pi}$ under the classical constraint $\bar{y}\bar{\pi} > 0$? We cannot simply ignore the constraint, since it is this that guarantees the positivity of $\bar{K}$. On top of this, the quantization of $\bar{K}$ is complicated by the factor $\operatorname{sgn}(\bar{y})$.

If we examine the classical system, we can see that it strongly suggests the partitioning of the lightcone into two disjoint regions $\mathscr{C}^{\pm}$ with $\mathscr{C}^{+}$ being the region on the right of the apex and $\mathscr{C}^{-}$ the regoin on the left, so that $\mathscr{C}^{-} \cup \mathscr{C}^{+} = \mathscr{C}_0$, which is the lightcone $\mathscr{C}$ with the apex removed. Everything is smooth and well defined within each of $\mathscr{C}^{\pm}$. If in the quantized theory we also relax the requirement of having global quantities and make some *ad hoc* quantizations, we can make some progress. First we construct the direct sum space

$$\mathscr{H}(\mathscr{C}_0)^{\oplus} = \mathscr{H}(\mathscr{C}^{-}) \oplus \mathscr{H}(\mathscr{C}^{+}), \qquad \text{where} \quad \mathscr{H}(\mathscr{C}^{\pm}) = L^2(\mathscr{C}^{\pm}, d\bar{y}/|\bar{y}|)$$

Next we can try to construct operators on $\mathscr{H}(\mathscr{C}^{-})$ and $\mathscr{H}(\mathscr{C}^{+})$ separately to correspond to observables on $\mathscr{C}^{+}$ and $\mathscr{C}^{-}$. Consider the operator

$$\hat{\Pi} = -i\hbar |\bar{y}|^{1/2} \frac{d}{d\bar{y}} \frac{1}{|\bar{y}|^{1/2}}$$

in $\mathscr{H}(\mathscr{C}^{+})$. With the usual domain this operator is maximal symmetric. Its absolute value $|\hat{\Pi}|$ is self-adjoint and positive (Weidmann, 1980). In passing we should point out that the operator $|\hat{\Pi}|$ has an undesirable feature, i.e., it possesses a degenerate spectrum. That aside, we may identify $|\hat{\Pi}|$ as the quantum observable corresponding to $\bar{\pi}$ in $\mathscr{C}^{+}$. It follows that one should take as the Hamiltonian in $\mathscr{H}(\mathscr{C}^{+})$ the positive operator

$$\hat{\bar{K}} = \frac{1}{2}\left(|\hat{\Pi}| + \frac{(m_0 c)^2}{|\hat{\Pi}|}\right)$$

Similar operators can be established on $\mathscr{H}(\mathscr{C}^{-})$.

We may try to go one step further and consider the relationship between these operators and those in the *instant form* as we did for the light front. On $\mathscr{C}^{+}$ the classical variables can be seen to be related by the same canonical transformation as that between the light-front and Cartesian coordinates (a different transformation has to be used on $\mathscr{C}^{-}$). However, we find that we cannot then proceed as before and obtain a unitary transform between the operators with the help of the paring construction.

There is an alternative approach based on a generalization of the maximal symmetric operator $\hat{\Pi}$ which can produce a positive-definite momentum operator without degeneracy in its spectrum; the idea is to introduce a Hilbert space on $\mathscr{C}_0$ in terms of two-component wave functions. A discussion of this would be deviating too far from our present study of quantum mechanics on the light front and so we shall not develop these ideas here.

9. CONCLUDING REMARKS

Geometric quantization furnishes a light-front quantum mechanics in a natural way. The adoption of generalized quantum mechanics makes it possible to accommodate the non-self-adjoint nature of the position operator. What is more, the pairing construction shows us how to relate the light-front and Cartesian pictures. In those instances where the same classical observable can be quantized in the light-front and in the Cartesian coordinate systems the resulting quantum observables are unitarily related by the pairing construction. Notice that this is not something that can be taken for granted. If a classical observable can be quantized geometrically in two canonically related coordinate systems and the two Hilbert spaces are related by a unitary map derived from the pairing, then it is not generally the case that the two representations of the quantized observable are related by that map. The fact that this is a property of the present scheme makes it all the more compelling.

APPENDIX A. ON CANONICAL TRANSFORMATION

First consider the simpler case of motion in $\mathscr{M}^2$. Suppose in light-front coordinates the particle is at position y with momentum π at light front "time" τ, and suppose that in Cartesian coordinates the position and momentum of the same particle are given by $x = q$ and p at Cartesian time $t = \tau$. Noting that $v_c = p/p_0$, it is easily verified (by examining the particle's motion in the space-time diagram of $\mathscr{M}^2$) that $x = q = y + y(p/p_0)$, or, equivalently, $y = qp_0/(p_0 + p)$. For the momentum variables one can easily verify the relationship $\pi = p_0 + p$. The coordinates (y, π) and (q, p) are therefore related by a canonical transformation since these relationships can be derived from the generating function given in the main text.

Next consider the general case of motion in $\mathscr{M}^4$. Then we have $v_c^1 = p_1/p_0$ and $v_c^s = p_s/p_0$, which lead to $q^1 = y^1 + y^1 p_1/p_0$ and $q^s = y^s + y^1 p_s/p_0$. The corresponding relationships between the canonical momenta are $\pi_1 = p_0 + p_1$, $\pi_s = p_s$. These describe a canonical transformation, since they are derivable from the generating function given in the main text.

APPENDIX B. ON MAXIMAL SYMMETRY OF $\hat{y}$

The operator $i\hbar\, d/dk$ in $\tilde{L}^2_+(\tilde{\mathscr{F}}_\tau, dk)$ is well known to be maximal symmetric (Akhiezer and Glazman, 1961, p. 111). It follows that its inverse Fourier transform $\hat{y}$ in $L^2_+(\mathscr{F}_\tau, dy)$ is maximal symmetric and so also is the operator $\hat{y}$ in $\mathscr{H}(\mathscr{F}_\tau)^+$.

APPENDIX C. ON THE POV MEASURE OF $\hat{y}$

Being maximal symmetric in $\mathscr{H}(\mathscr{F}_\tau)^+$, the operator $\hat{y}$ possesses a unique POV measure (Akhiezer and Glazman, 1963, p. 135). Since $\hat{y}$ in $\mathscr{H}(\mathscr{F}_\tau)^+$ is the reduction of $\hat{y}$ in $\mathscr{H}(\mathscr{F}_\tau)$ to $\mathscr{H}(\mathscr{F}_\tau)^+ = \hat{R}\mathscr{H}(\mathscr{F}_\tau)$ and since the spectral measure of $\hat{y}$ in $\mathscr{H}(\mathscr{F}_\tau)$ is given by χ_b, where b are the Borel sets of the reals, the required POV measure is simply $\hat{P}(\hat{y}; b) = \hat{R}\chi_b\hat{R}$ by Naimark's theorem (Akhiezer and Glazman, 1963, p. 121).

APPENDIX D. ON PROBABILITY AND PROBABILITY CURRENT DENSITIES

If we put

$$\rho_0 = \frac{|\Phi|^2}{|y|}, \qquad \rho_1 = \frac{1}{2}\left\{\frac{|\Phi|^2}{|y|} + \left(\frac{m_0^2 c^2}{|y|^{1/2}}\right)\hat{\pi}^{-1}\left(\frac{(\hat{\pi}^{-1}\Phi^*)\Phi + \Phi^*(\hat{\pi}^{-1}\Phi)}{|y|^{1/2}}\right)\right\}$$

then the continuity equation

$$\frac{\partial\rho_0}{\partial t} + \frac{\partial\rho_1}{\partial y} = 0$$

follows from the Schrödinger equation

$$i\hbar\frac{\partial\Phi}{\partial t} = \hat{K}\Phi$$

The corresponding expressions in $L_+^2(\mathscr{F}_\tau, dy)$ are

$$\rho_0 = |\phi|^2, \qquad \rho_1 = \frac{c}{2}\{|\phi|^2 + m_0^2 c^2 \hat{k}_+^{-1}[(\hat{k}_+^{-1}\phi^*)\phi + \phi^*(\hat{k}_+^{-1}\phi)]\}$$

The expressions for ρ_1 are useful only when $\hat{\pi}^{-1}$ and $\hat{k}_+^{-1}$ can be meaningfully applied.

In $\mathscr{M}^4$ there is also a continuity equation with

$$\rho_0 = \frac{|\Phi|^2}{|y^1|}, \qquad \rho_1 = \frac{1}{2}\left\{\frac{|\Phi|^2}{|y^1|} + \frac{m^2c^2}{|y^1|^{1/2}}\hat{\pi}_1^{-1}[(\hat{\pi}_1^{-1}\Phi^*)\Phi + \Phi^*(\hat{\pi}_1^{-1}\Phi)]\right\}$$

and

$$\rho_s = \frac{1}{2}\hat{\pi}_s^{-1}\left(\frac{\Phi^*(\hat{\pi}_s^2\hat{\pi}_1^{-1}\Phi) + (\hat{\pi}_s^2\hat{\pi}_1^{-1}\Phi^*)\Phi}{|y^1|}\right)$$

APPENDIX E. ON THE UNITARY TRANSFORMATION T

For each $\Phi(y) \in \mathscr{H}(\mathscr{F}_\tau)^+$ we have $\phi(y) = \Phi(y)/|y|^{1/2} = (V^{-1}\Phi)(y) \in L^2_+(\mathscr{F}_\tau, dy)$ and its Fourier transform $\tilde{\phi}(k)$ which vanishes for $k < 0$. It follows that

$$\|\Phi\|^2 = \|\phi\|^2 = \int_0^\infty |\tilde{\phi}(k)|^2 \, dk$$

Since $\tilde{\psi}(p) = (T\Phi)(p) = \tilde{\phi}(p + p_0)[(p + p_0)/p_0]^{1/2}$ we have

$$\begin{aligned}\|\tilde{\psi}\|^2 &= \int_{-\infty}^\infty |\tilde{\psi}(p)|^2 \, dp = \int_{-\infty}^\infty |\tilde{\phi}(p + p_0)|^2 \left(\frac{p + p_0}{p_0}\right) dp \\ &= \int_0^\infty |\tilde{\phi}|^2(u) \, du = \|\tilde{\phi}\|^2 = \|\Phi\|^2\end{aligned}$$

Next, we have

$$\begin{aligned}T^{-1}T\Phi &= \frac{|y|^{1/2}}{(2\pi\hbar)^{1/2}} \int_{-\infty}^\infty \tilde{\phi}(p + p_0)\left(\frac{p + p_0}{p_0}\right)^{1/2} \\ &\quad \times \exp[-\not{i}(p + p_0)y]\left(\frac{p + p_0}{p_0}\right)^{1/2} dp \\ &= \frac{|y|^{1/2}}{(2\pi\hbar)^{1/2}} \int_0^\infty \tilde{\phi}(k) \exp(-\not{i}ky) \, dk = \Phi\end{aligned}$$

Finally, for each $\tilde{\psi}(p) \in \mathscr{H}(\mathscr{S}_t)$ define a function $\tilde{\psi}'(k)$ of $k = p + p_0$ by $\tilde{\psi}'(k) = [p_0/(p + p_0)]^{1/2}\tilde{\psi}(p)$. Then

$$\int_0^\infty |\tilde{\psi}'(k)|^2 \, dk = \int_{-\infty}^\infty |\tilde{\psi}(p)|^2 \, dp < \infty$$

We can identify $\tilde{\psi}'$ with $\tilde{\phi} \in L^2_+(\mathscr{F}_\tau, dy)$. Consequently each $\tilde{\psi} \in \mathscr{H}(\mathscr{S}_t)$ is mapped to $\Phi \in \mathscr{H}(\mathscr{F}_\tau)^+$ via $\tilde{\phi}$ and $\tilde{\psi}'$ by

$$\begin{aligned}\Phi(y) = V\phi &= \frac{|y|^{1/2}}{(2\pi\hbar)^{1/2}} \int_0^\infty \tilde{\phi}(k) \exp(\not{i}ky) \, dk \\ &= \frac{|y|^{1/2}}{(2\pi\hbar)^{1/2}} \int_0^\infty \tilde{\psi}'(k) \exp(\not{i}ky) \, dk \\ &= \frac{|y|^{1/2}}{(2\pi\hbar)^{1/2}} \int_{-\infty}^\infty \tilde{\psi}(p) \exp[\not{i}(p + p_0)y]\left(\frac{p + p_0}{p_0}\right)^{1/2} dp = (T^{-1}\tilde{\psi})(y)\end{aligned}$$

These results confirm the unitary nature of T.

APPENDIX F. ON MAXIMAL SYMMETRY AND SELF-ADJOINTNESS OF TENSOR PRODUCT OPERATORS

First we can identify $L^2(\mathscr{F}_\tau, d^3y)$ with the tensor product space $L^2(\mathscr{F}_\tau, dy^1) \otimes L^2(\mathbb{R}^2, dy^2\, dy^3)$. The maximal symmetry of $\hat{y}^1$ follows from the same property of the corresponding operator in $L^2(\mathscr{F}_\tau, dy^1)$ because of the proposition given below.

Let A_1 be a maximal symmetric operator in a Hilbert space $\mathscr{H}_1$ with deficiency indices $n_+ = 1, n_- = 0$. Let $\mathbb{I}_2$ be the identity on Hilbert space $\mathscr{H}_2$. Let $\mathscr{H}_c = \mathscr{H}_1 \otimes \mathscr{H}_2$ and $A_c = A_1 \otimes \mathbb{I}_2$. Then we have the following result:

Proposition. A_c is maximal symmetric in $\mathscr{H}_c$ with deficiency indices $n_{c+} = \dim\{\mathscr{H}_2\}$ and $n_{c-} = 0$.

Proof. Suppose $\mathscr{H}_2$ is one-dimensional. Then the deficiency indices of A_c clearly coincide with those of A_1. Next suppose $\mathscr{H}_2$ is of finite dimension m. Let $\{\mathbb{I}_{2j}, j = 1, 2, \ldots, m\}$ be a decomposition of the identity $\mathbb{I}_2$ in $\mathscr{H}_2$. Then we can write

$$A_c = A_1 \otimes \mathbb{I}_2 = \bigoplus_{j=1}^{m} (A_1 \otimes \mathbb{I}_{2j})$$

It follows that $n_{c+} = m$ and $n_{c-} = 0$ since the deficiency indices of the direct sum operator A_c are equal to the sums of the corresponding indices of the operators $A_1 \otimes \mathbb{I}_{2j}$ (Akhiezer and Glazman, 1963, p. 128). This result also holds when $\mathscr{H}_2$ is infinite-dimensional: in such a case the result will be $n_{c+} = \infty$ and $n_{c-} = 0$. In all these cases A_c is maximal symmetric. For a similar proof see Exner and Seba (1987).

Finally many of the operators in the main text can be identified with the closures of tensor products of self-adjoint operators. It then follows that these operators are self-adjoint (Wiedmann, 1980).

APPENDIX G. ON THE POINCARÉ ALGEBRA

Define the classical Poisson bracket by

$$\{G, E\} = \sum_j \left(\frac{\partial G}{\partial q^j}\frac{\partial E}{\partial p_j} - \frac{\partial E}{\partial q^j}\frac{\partial G}{\partial p_j}\right)$$

Then the ten generators satisfy the following standard Poisson bracket relations:

$$\{P_\mu, P_\nu\} = 0, \qquad \{J_{\kappa\lambda}, P_\nu\} = \eta_{\nu\kappa}P_\lambda - \eta_{\nu\lambda}P_\kappa$$

$$\{J_{\mu\nu}, J_{\lambda\sigma}\} = \eta_{\lambda\nu}J_{\sigma\mu} + \eta_{\nu\sigma}J_{\mu\lambda} + \eta_{\sigma\mu}J_{\lambda\nu} + \eta_{\mu\lambda}J_{\nu\sigma}$$

A lot of work is required to verify the completeness of the vector fields generated by these classical observables. Here we shall illustrate the calculations involved by considering the completeness of the vector field

$$X_{31} = -\pi_3 \frac{\partial}{\partial \pi_1} + \left(\frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1} \right) \frac{\partial}{\partial \pi_3}$$

generated by J_{31}. The integral curves of X_{31} satisfy the following equations (r a parameter):

$$\frac{d\pi_1}{dr} = -\pi_3, \qquad \frac{d\pi_2}{dr} = 0, \qquad \frac{d\pi_3}{dr} = \left(\frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (m_0 c)^2}{2\pi_1} \right)$$

The integral curve originating at the point $\boldsymbol{\pi} = \boldsymbol{\pi}_0$ at $r = 0$ is given by

$$\begin{aligned} \pi_1(r) &= [(p_1^0)^2 + (\pi_3^0)^2]^{1/2} \sin(r + \alpha^0) + p_0^0 \\ \pi_2(r) &= \pi_2^0 \\ \pi_3(r) &= [(p_1^0)^2 + (\pi_3^0)^2]^{1/2} \cos(r + \alpha^0) \\ \alpha^0 &= \arctan\left(\frac{p_0^0 - \pi_1^0}{\pi_3^0} \right) \end{aligned}$$

Here, p_1 and p_0 are regarded as functions of π_j; p_1^0 and p_0^0 are the values of p_1 and p_0 corresponding to $\pi_j = \pi_j^0$ at $r = 0$. Note that p_0 remains constant along the integral curve, i.e., $dp_0/dr = 0$. The above solutions are well defined for all $r \in \mathbb{R}$ and are consistent with the constraint $\pi_1 > 0$ since $p_0^0 > [(p_1^0)^2 + (\pi_3^0)^2]^{1/2}$. It follows that the vector field X_{31} is complete. The quantized operator J_{31} will then be essentially self-adjoint on the set of infinitely differentiable functions of compact support on $\tilde{\mathscr{F}}_\tau^+$ and so will have a unique self-adjoint extension (Wan and Viasminsky, 1977; Mackey, 1963, pp. 100–104).

In passing we should mention that the vector field X_{31} has critical points; this requires some careful consideration. We shall not pursue this here since a similar problem was discussed in detail by Wan and Viasminsky (1977).

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REFERENCES

Akhiezer, N. I., and Glazman, I. M. (1961). *Theory of Linear Operators in Hilbert Space*, Vol. 1, Ungar, New York.

Akhiezer, N. I., and Glazman, I. M. (1963). *Theory of Linear Operators in Hilbert Space*, Vol. 2, Ungar, New York.

Bohm, A., and Gadella, M. (1989). *Dirac Kets, Gamow Vectors and Gelfand Triplets*, Springer-Verlag, New York.

Busch, P., Lahti, P. J., and Mittelstaedt, P. (1991). *The Quantum Theory of Measurement*, Springer-Verlag, New York.

Chang, S.-J., Root, R. G., and Yan, T.-M. (1973). *Physical Reviews D*, **7**, 1133.

Derrick, G. H. (1987*a*). *Journal of Mathematical Physics*, **28**, 64.

Derrick, G. H. (1987*b*). *Journal of Mathematical Physics*, **28**, 1327.

Dirac, P. A. M. (1949). *Reviews of Modern Physics*, **21**, 392.

Dirac, P. A. M. (1966). *Belfer Lectures on Quantum Field Theory*, Vols. 1 and 2; Belfer Graduate School of Science, Yeshiva University, New York.

Exner, P., and Seba, P. (1987). *Journal of Mathematical Physics*, **28**, 386.

Harindranath, A., and Vary, J. P. (1987). *Physical Review D*, **36**, 1141.

Hegerfeldt, G. C. (1974). *Physical Review D*, **10**, 3320.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kalnay, A. J. (1971). In *Problems in the Foundations of Physics*, M. Bunge, ed., Springer-Verlag, New York.

Kraus, K. (1983). *States, Effects, and Operations*, Springer-Verlag, New York.

Mackey, G. W. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, Menlo Park, California.

Mosley, S. N., and Farina, J. E. G. (1992*a*). *Journal of Physics A*, **25**, 4673.

Mosley, S. N., and Farina, J. E. G. (1992*b*). *Journal of Physics A*, **25**, 4687.

Pauli, H.-C., and Brodsky, J. (1985). *Physical Review D*, **32**, 1993.

Peres, A. (1967). *Journal of Mathematical Physics*, **9**, 785.

Prugovecki, E. (1986). *Stochastic Quantum Mechanics and Quantum Spacetime*, Reidel, Boston.

Schweber, S. S. (1964). *An Introduction of Relativistic Quantum Field Theory*, Harper & Row, New York, and Weatherhill, Tokyo, Chapter 3.

Wan, K. K., and Viasminsky, C. (1977). *Progress of Theoretical Physics*, **58**, 1030.

Wan, K. K., Jackson, T., and McKenna, I. (1984*a*). *Nuovo Cimento B*, **81**, 165.

Wan, K. K., McKenna I., and Pinto, J. (1984*b*). *Algebras Groups Geometries*, **1**, 344.

Weidmann, J. (1980). *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York.

Woodhouse, N. (1980). *Geometric Quantization*, Clarendon Press, Oxford.